Department of Mathematics – University of Trento

Algorithms for rank and cactus rank of a polynomial

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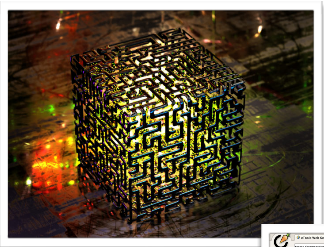
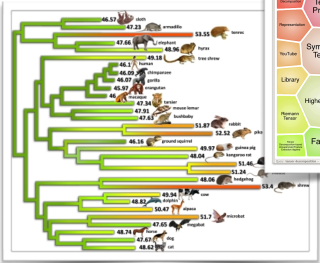
Outline

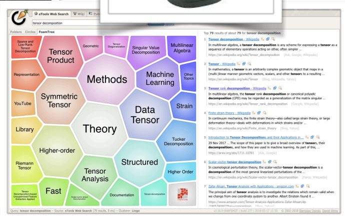
[BCMT] Symmetric Tensor Decomposition, 2010

J. Brachat, P. Comon, B. Mourrain and E. Tsigaridas.
Linear Algebra and its Applications, 433 (11–12), pp. 1851-1872.

Applications







You have...

$$F = -4xy + 2xz + 2yz + z^2.$$

$$\Downarrow$$

You want...

$$F = (x - y)^2 - 2(x + y)^2 + (x + y + z)^2.$$

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$$F = -4xy + 2xz + 2yz + z^2,$$
$$f = F_{x=1} = -4y + 2z + 2yz + z^2.$$

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You want...

$$F = (x-y)^2 - 2(x+y)^2 + (x+y+z)^2.$$
$$f = F_{x=1} = (1-y)^2 - 2(1+y)^2 + (1+y+z)^2.$$

Ideas

Move and solve the problem in the dual space

$$R = \mathbb{K}[x_1, \ldots, x_n].$$

Apolar polynomial

$$f = \sum_{|\alpha| \le d} f_\alpha \mathbf{x}^\alpha \in R_{\le d}$$

$$\downarrow$$

$$f^* : R_{\le d} \to \mathbb{K},$$

$$g = \sum_{|\alpha| \le d} g_\alpha \mathbf{x}^\alpha \mapsto \langle f, g \rangle = \sum_{|\alpha| \le d} \frac{f_\alpha g_\alpha}{\binom{d}{\alpha}}$$

Ideas

Move and solve the problem in the dual space

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$$f^* = \Big(\sum_{|\alpha|\le d} f_\alpha \mathbf{x}^\alpha\Big)^* : R_{\le d} \to \mathbb{K},$$

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Dual map

$$\tau : R_{\le d} \hookrightarrow R^*_{\le d},$$

$$f = \sum_{i=1}^{r} \lambda_i (1 + l_{1i}x_1 + \cdots + l_{ni}x_n)^d \mapsto f^* = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{(l_{1i},\ldots,l_{ni})}.$$

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Aim

Find $\Lambda \in R^*$ that restricts to f^* on $R_{\leq d}$:

$$\Lambda|_{R_{\leq d}} = f^*.$$

Ideas
Use Henkel operators

Let $\Lambda \in R^*$. Define

- the Henkel operator of Λ as

$$\begin{aligned} H_\Lambda : R &\to R^*, \\ r &\mapsto r \star \Lambda = \big(t \mapsto \Lambda(rt)\big), \end{aligned}$$

- $I_\Lambda = \ker H_\Lambda$,
- $\mathcal{A}_\Lambda = R/I_\Lambda$,
- the multiplication by r operators on $\mathcal{A}_\Lambda$ and $\mathcal{A}_\Lambda^*$ as

$$\begin{aligned} M_r : \mathcal{A}_\Lambda &\to \mathcal{A}_\Lambda, \\ t &\mapsto r \cdot t, \end{aligned} \qquad \begin{aligned} M_r^t : \mathcal{A}_\Lambda^* &\to \mathcal{A}_\Lambda^*, \\ \phi &\mapsto r \star \phi. \end{aligned}$$

[BCMT] Theorem

Let $\Lambda \in R^*$ and $r \in \mathbb{N}_{>0}$. The following are equivalent:

- There exist non-zero constants $\{\lambda_i\}_{i\in\{1,\dots,r\}}$ and distinct points $\{\zeta_i\}_{i\in\{1,\dots,r\}} \subseteq \mathbb{K}^n$ such that

$$\Lambda = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{\zeta_i}.$$

- $\mathrm{rk} H_\Lambda = r$ and I_Λ is a radical ideal.

Ideas

Use Henkel operators

Theorem

Let $\Lambda \in R^*$ such that I_Λ is 0-dimensional and $\mathcal{A}_\Lambda$ is an r-dimensional $\mathbb{K}$-vector space. Then the following are equivalent:

- Up to $\mathbb{K}$-multiplication, there are r distinct common eigenvectors of $\{M_{x_i}^t\}_{i\in\{1,\dots,n\}}$.
- I_Λ is radical.

Theorem

Let $\Lambda \in R^*$ such that I_Λ is 0-dimensional and $\mathcal{A}_\Lambda$ is an r-dimensional $\mathbb{K}$-vector space.

- $\mathcal{V}(I_\Lambda) = \{\zeta_1, \ldots, \zeta_s\}$ is radical if and only if $s = r$ since $\mathcal{A}_\Lambda = R/I_\Lambda$ and $\dim(\mathcal{A}_\Lambda) = r$.

Then the following are equivalent:

- Up to $\mathbb{K}$-multiplication, there are r distinct common eigenvectors of $\{M^t_{x_i}\}_{i \in \{1,\ldots,n\}}$.
  - Eigenvalues of M_{x_i} and $M^t_{x_i}$ are $\{x_i(\zeta_1), \ldots, x_i(\zeta_s)\}$. [Stickelberger]
  - v is an eigenvector for every $\{M^t_{x_i}\}_{i \in \{1,\ldots,n\}}$ if and only if there exist $\zeta_1, \ldots, \zeta_s \in \mathbb{K}^n$ and $k \neq 0$ such that $v = k\mathbb{1}_{\zeta_j}$.
- I_Λ is radical.

Ideas
Fill the Henkel matrix

Let $f = -4y + 2z + 2yz + z^2$.
We know some entries of $\mathbb{H}_\Lambda$:

$$\mathbb{H}_\Lambda = \left(\begin{array}{c|cccccc}
 & 1 & y & z & y^2 & yz & z^2 \\ \hline
1 & f^*(1) & f^*(y) & f^*(z) & f^*(y^2) & f^*(yz) & f^*(z^2) \\
y & f^*(y) & f^*(y^2) & f^*(yz) & & & \\
z & f^*(z) & f^*(yz) & f^*(z^2) & & & \\
y^2 & f^*(y^2) & & & & & \\
yz & f^*(yz) & & & & ? & \\
z^2 & f^*(z^2) & & & & &
\end{array}\right).$$

Let $f = -4y + 2z + 2yz + z^2$.

$$\mathbb{H}_\Lambda(\mathbf{h}) = \left(\begin{array}{c|cccccc} & 1 & y & z & y^2 & yz & z^2 \\ \hline 1 & 0 & -2 & 1 & 0 & 1 & 1 \\ y & -2 & 0 & 1 & h_{(3,0)} & h_{(2,1)} & h_{(1,2)} \\ z & 1 & 1 & 1 & h_{(2,1)} & h_{(1,2)} & h_{(0,3)} \\ y^2 & 0 & h_{(3,0)} & h_{(2,1)} & h_{(4,0)} & h_{(3,1)} & h_{(2,2)} \\ yz & 1 & h_{(2,1)} & h_{(1,2)} & h_{(3,1)} & h_{(2,2)} & h_{(1,3)} \\ z^2 & 1 & h_{(1,2)} & h_{(0,3)} & h_{(2,2)} & h_{(1,3)} & h_{(0,4)} \end{array}\right).$$

We want values for **h** such that $\mathrm{rk} H_\Lambda = r$ and I_Λ is radical.

Let $f = -4y + 2z + 2yz + z^2$.

$$\mathbb{H}_\Lambda(\mathbf{h}) = \begin{array}{c|cccccc} & 1 & y & z & y^2 & yz & z^2 \\ \hline 1 & 0 & -2 & 1 & 0 & 1 & 1 \\ y & -2 & 0 & 1 & h_{(3,0)} & h_{(2,1)} & h_{(1,2)} \\ z & 1 & 1 & 1 & h_{(2,1)} & h_{(1,2)} & h_{(0,3)} \\ y^2 & 0 & h_{(3,0)} & h_{(2,1)} & h_{(4,0)} & h_{(3,1)} & h_{(2,2)} \\ yz & 1 & h_{(2,1)} & h_{(1,2)} & h_{(3,1)} & h_{(2,2)} & h_{(1,3)} \\ z^2 & 1 & h_{(1,2)} & h_{(0,3)} & h_{(2,2)} & h_{(1,3)} & h_{(0,4)} \end{array}.$$

We guess that $B = \{1, y, z\}$ is a basis for $\mathcal{A}_\Lambda$, so that $r = 3$. Define

$$\mathbb{H}_\Lambda^B = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

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$$\mathbb{H}_\Lambda(\mathbf{h}) = \begin{array}{c|cccccc} & 1 & y & z & y^2 & yz & z^2 \\ \hline 1 & 0 & -2 & 1 & 0 & 1 & 1 \\ y & -2 & 0 & 1 & h_{(3,0)} & h_{(2,1)} & h_{(1,2)} \\ z & 1 & 1 & 1 & h_{(2,1)} & h_{(1,2)} & h_{(0,3)} \\ y^2 & 0 & h_{(3,0)} & h_{(2,1)} & h_{(4,0)} & h_{(3,1)} & h_{(2,2)} \\ yz & 1 & h_{(2,1)} & h_{(1,2)} & h_{(3,1)} & h_{(2,2)} & h_{(1,3)} \\ z^2 & 1 & h_{(1,2)} & h_{(0,3)} & h_{(2,2)} & h_{(1,3)} & h_{(0,4)} \end{array}.$$

We guess that $B = \{1, y, z\}$ is a basis for $\mathcal{A}_\Lambda$, so that $r = 3$. Define

$$\mathbb{H}^B_\Lambda = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbb{H}^B_{y\star\Lambda} = \begin{pmatrix} -2 & 0 & 1 \\ 0 & h_{(3,0)} & h_{(2,1)} \\ 1 & h_{(2,1)} & h_{(1,2)} \end{pmatrix}.$$

Let $f = -4y + 2z + 2yz + z^2$.
We guess that $B = \{1, y, z\}$ is a basis for $\mathcal{A}_\Lambda$, so that $r = 3$. Define

$$\mathbb{M}_y^B = \mathbb{H}_{y\star\Lambda}^B (\mathbb{H}_\Lambda^B)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{3}{8} h_{(3,0)} + \frac{1}{4} h_{(2,1)} & \frac{1}{8} h_{(3,0)} + \frac{1}{4} h_{(2,1)} & \frac{1}{4} h_{(3,0)} + \frac{1}{2} h_{(2,1)} \\ -\frac{3}{8} h_{(2,1)} + \frac{1}{4} h_{(1,2)} + \frac{1}{8} & \frac{1}{8} h_{(2,1)} + \frac{1}{4} h_{(1,2)} - \frac{3}{8} & \frac{1}{4} h_{(2,1)} + \frac{1}{2} h_{(1,2)} + \frac{1}{4} \end{pmatrix},$$

$$\mathbb{M}_z^B = \mathbb{H}_{z\star\Lambda}^B (\mathbb{H}_\Lambda^B)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{3}{8} h_{(2,1)} + \frac{1}{4} h_{(1,2)} + \frac{1}{8} & \frac{1}{8} h_{(2,1)} + \frac{1}{4} h_{(1,2)} - \frac{3}{8} & \frac{1}{4} h_{(2,1)} + \frac{1}{2} h_{(1,2)} + \frac{1}{4} \\ -\frac{3}{8} h_{(1,2)} + \frac{1}{4} h_{(0,3)} + \frac{1}{8} & \frac{1}{8} h_{(1,2)} + \frac{1}{4} h_{(0,3)} - \frac{3}{8} & \frac{1}{4} h_{(1,2)} + \frac{1}{2} h_{(0,3)} + \frac{1}{4} \end{pmatrix}.$$

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We want multiplication operators to commute!

$$\mathbb{M}_y^B \mathbb{M}_z^B - \mathbb{M}_z^B \mathbb{M}_y^B = 0.$$

$$\to h_{(3,0)} = -2,\ h_{(2,1)} = 1,\ h_{(2,1)} = 1,\ h_{(2,1)} = 4.$$

Let $f = -4y + 2z + 2yz + z^2$.

$$(\mathbb{M}_y^B)^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \text{Eigenspaces: } \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

$$(\mathbb{M}_z^B)^t = \begin{pmatrix} 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{3}{4} \\ 1 & 1 & \frac{5}{2} \end{pmatrix} \rightarrow \text{Eigenspaces: } \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\rangle.$$

Let $f = -4y + 2z + 2yz + z^2$.

Common eigenspaces: $\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\rangle$.

Solve in λ_i : $f = \lambda_1 \left(1 - 1y + 0z\right)^2 + \lambda_2 \left(1 + 1y - \frac{1}{2}z\right)^2 + \lambda_3 \left(1 + 1y + 3z\right)^2$.

$\lambda_1 = 1$ $\lambda_2 = -\frac{8}{7}$ $\lambda_3 = \frac{1}{7}$

Conclusion: $f = \left(1 - y\right)^2 - \frac{8}{7}\left(1 + y - \frac{1}{2}z\right)^2 + \frac{1}{7}\left(1 + y + 3z\right)^2$.

Algorithm: Symmetric tensor decomposition
Input: A homogeneous polynomial $F(x_0, x_1, \dots, x_n)$ of degree d.
Output: A decomposition of F as $F = \sum_{i=1}^{r} \lambda_i L_i^d$ with r minimal.

- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \leq d$.
- $r := 1$.
- **repeat**
  1. Compute a set B of monomials of degree at most d connected to one with $|B| = r$.
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i \star \Lambda}^B (\mathbb{H}_\Lambda^B)^{-1}$ commute.
  3. If there is no solution, restart the loop with $r := r + 1$.
  4. Else compute the $n \times r$ eigenvalues $\zeta_{i,j}$ and the eigenvectors $\mathbf{v_j}$ s.t. $\mathbb{M}_i \mathbf{v_j} = \zeta_{i,j} \mathbf{v_j}$, $i = 1, \dots, n, j = 1, \dots, r$.

  until the eigenvalues are simple.
- Solve the linear system in $(l_j)_{j=1,\dots,k}$: $\Lambda = \sum_{i=1}^{r} l_j \mathbb{1}_{\zeta_j}$ where $\zeta_i \in \mathbb{K}^n$ are the eigenvectors found in step 4.

Algorithm: Symmetric tensor decomposition
Input: A homogeneous polynomial $F(x_0, x_1, \dots, x_n)$ of degree d
written by using a general set of essential variables.
Output: A decomposition of f as $F = \sum_{i=1}^{r} \lambda_i \mathbf{k_i}(\mathbf{x})^d$ with r minimal.

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- General: De-homog. by x_0 implies decomp. of type

$$f = \sum_{i=1}^{r} \lambda_i \big(1 + \alpha_i l_i(x_1, \ldots, x_n)\big)^d$$

- Essential variables:

$$x^3 + (x + y + z)^3$$

2 essential variables and rank 2 $\Rightarrow$ Any basis made of 2 elements
$\Rightarrow$ we can recover at most 2 coefficients of the linear forms.

The refinements

1) The starting r

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- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \le d$.
- ~~$r := 1$~~. r := #EssVar(f)?
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- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \leq d$.
- $\cancel{r := 1}$. r := rk(Maximal numerical submatrix of H_Λ).
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  1. Compute a set B of monomials of degree at most d connected to one with $|B| = r$.
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- ~~$r := 1$~~. r := rk(Maximal numerical submatrix of H_Λ).

1. (I.K.) $\rightsquigarrow$ (rk(Maximal numerical submatrix of H_Λ))$\leq rk(F)$
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2. Not only a matter of time consuming:

$$F = x^4 + (x+y)^4 + (x-y)^4 = 3x^4 + 12x^2y^2 + 2y^4$$

$$\begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & h_5 \\ 0 & 2 & h_5 & h_6 \end{pmatrix}$$

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Start with $r = 2$ (instead of $r = 3$). Only one basis: $B = \{1, y\}$.

$$\mathbb{M}_y^B = \begin{pmatrix} 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix} \text{ has two eigenvectors } (\pm\sqrt{3/2}, 1)$$

but the system $F = \lambda_1(\sqrt{3/2}x + y)^3 + \lambda_2(-\sqrt{3/2}x + y)^3$ has no solutions.

- ~~$r := 1$~~. r := rk(Maximal numerical submatrix of H_Λ).

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$$\mathbb{M}_y^B = \begin{pmatrix} 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix} \text{ has two eigenvectors } (\pm\sqrt{3/2}, 1)$$

but the system $F = \lambda_1(\sqrt{3/2}x + y)^3 + \lambda_2(-\sqrt{3/2}x + y)^3$ has no solutions. Ignore the condition imposed by the coefficients of y^3, then system has solution, that is $\lambda_1 = \lambda_2 = \frac{2}{3}$ which lead to $G = 3x^4 + 12x^2y^2 + \frac{4}{3}y^4$.

The refinements

2) Connection to one vs staircases

Algorithm: Symmetric tensor decomposition

Input: A homogeneous polynomial $F(x_0, x_1, \ldots, x_n)$ of degree d.
written by using a general set of essential variables.

Output: A decomposition of F as $F = \sum_{i=1}^{r} \lambda_i \mathbf{k_i}(\mathbf{x})^d$ with r minimal.

- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \leq d$.
- r := rk(largest numerical submatrix of H_Λ).
- **repeat**
  1. Compute a set B of monomials of degree at most d ~~connected to one~~ which is a complete staircase with $|B| = r$.
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i \star \Lambda}^B (\mathbb{H}_\Lambda^B)^{-1}$ commute.
  3. If there is no solution, restart the loop with $r := r + 1$.
  4. Else compute the $n \times r$ eigenvalues $\zeta_{i,j}$ and the eigenvectors $\mathbf{v_j}$ s.t. $\mathbb{M}_i \mathbf{v_j} = \zeta_{i,j} \mathbf{v_j}$, $i = 1, \ldots, n, j = 1, \ldots, r$.

  until the eigenvalues are simple.
- Solve the linear system in $(l_j)_{j=1,\ldots,k}$: $\Lambda = \sum_{i=1}^{r} l_j \mathbb{1}_{\zeta_j}$ where $\zeta_i \in \mathbb{K}^n$ are the eigenvectors found in step 4.

Connection to one: $B = \{1, y, y^2, y^2z, y^3\}$.
Complete staircase: $B = \{1, y, z, y^2, yz\}$.

Theorem

Let $F \in R$ be homogeneous written by using essential variables and let $\Lambda \in R^*$ be an extension of $f^* \in R^*_{\leq d}$. Then there is a monomial basis B of $\mathcal{A}_\Lambda$ such that B is a complete staircase.

Comparison with 3 variables

Size of B	# Complete staircases	# Connected to 1
3	1	5
4	3	13
5	5	35
6	9	96
7	13	267

Algorithm: Symmetric tensor decomposition

Input: A homogeneous polynomial $F(x_0, x_1, \dots, x_n)$ of degree d.

written by using a general set of essential variables.

Output: A decomposition of F as $F = \sum_{i=1}^{r} \lambda_i \mathbf{k_i}(\mathbf{x})^d$ with r minimal.

- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \le d$.
- r := rk(largest numerical submatrix of H_Λ).
- **repeat**
  1. Compute a set B of monomials of degree at most d which is a complete staircase with $|B| = r$.
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i \star \Lambda}^B (\mathbb{H}_\Lambda^B)^{-1}$ commute.
  3. If there is no solution, restart the loop with $r := r + 1$.
  4. Else compute the $n \times r$ eigenvalues $\zeta_{i,j}$ and the eigenvectors $\mathbf{v_j}$ s.t. $\mathbb{M}_i \mathbf{v_j} = \zeta_{i,j} \mathbf{v_j}$, $i = 1, \dots, n, j = 1, \dots, r$.

  until ~~the eigenvalues are simple.~~ there are r common eigenvectors.
- Solve the linear system in $(l_j)_{j=1,\dots,k}$: $\Lambda = \sum_{i=1}^{r} l_j \mathbb{1}_{\zeta_j}$ where $\zeta_i \in \mathbb{K}^n$ are the eigenvectors found in step 4.

Example

$$F = (x+y)^3 + (x+z)^3 + (x+y+z)^3$$

$$\downarrow$$

$$\mathbb{M}_y^B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad \mathbb{M}_z^B = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\downarrow$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle.$$

What we can learn more

Let $F := (x+y)^5 + (x+z)^5 + (x+2y)(x-y)^4$.
We check $r = 4$ and $B = \{1, y, z, y^2\}$.

$$\mathbb{M}_y^B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad \mathbb{M}_z^B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\downarrow$$

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What we can learn more

Tangential decomposition

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$$\downarrow$$

$$\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \end{pmatrix} \right\rangle, \leftarrow \text{Generalized!}$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

What we can learn more

Tangential decomposition

Definition

The *tangential rank* of F is the minimal $r \in \mathbb{N}$ such that

$$F = \sum_{i=1}^{k} \lambda_i L_i^{d-1} L_{s+i} + \sum_{i=k+1}^{s} \lambda_i L_i^d$$

with $k + s = r$ and $k \leq s$. Such a decomposition for which r is minimal is a *tangential decomposition* of F.

What we can learn more
Tangential decomposition

Proposition

Let $l = 1 + l_1x_1 + \cdots + l_nx_n \in R_{\leq 1}$ and $g = 1 + g_1x_1 + \cdots + g_nx_n \in R_{\leq 1}$. For every $d \in \mathbb{Z}_{\geq 1}$ we have

$$\tau(l^{d-1}g) = \mathbb{1}_l + \frac{1}{d}\mathbb{1}_l \circ \left[\sum_{i=1}^{n}(g_i - l_i)\frac{\partial}{\partial x_i}\right] \in R^*_{\leq d}.$$

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Theorem

Let $\Lambda \in R^*$ be an extension of $f^* \in R^*_{\leq d}$ with $F = L^{d-1} G$. Then

- The common eigenvector of $M^t_{x_j}$ is $\mathbb{1}_l$.
- The generalized rank-2 eigenvector of each $M^t_{x_j}$ is $\mathbb{1}_l \circ \left[\sum_{i=1}^{n} (g_i - l_i) \frac{\partial}{\partial x_i} \right]$.

Input: A homogeneous polynomial $F(x_0, x_1, \ldots, x_n)$ of degree d.
Output: A minimal decomposition of F as
$F = \sum_{i=1}^{k} \lambda_i L_i^{d-1} L_{s+i} + \sum_{i=k+1}^{s} \lambda_i L_i^{d}$.

- Construct the matrix $\mathbb{H}_\Lambda(\mathbf{h})$ with the parameters $\mathbf{h} = \{h_\alpha\}_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|>d}}$.
- Set $r := \mathrm{rk}\mathbb{H}_{f^*}^{\square}$.
- **repeat**
  1. Compute a set B of a complete staircase monomials with $|B| = r$.
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i\star\Lambda}^B (\mathbb{H}_\Lambda^B)^{-1}$ commute.
  3. If there is no solution, restart the loop with $r := r + 1$.
  4. Else compute the $\frac{r}{2} \leq s \leq r$ eigenvectors of a generic $\sum_i \alpha_i \mathbb{M}_i^B$.

  until There are $r - s$ distinct generalized of rank up to 2 eigenvectors $v_{s+1}, \ldots, v_r$ common to $\mathbb{M}_i^B$'s such that
  - they have rank 2 for at least one $\mathbb{M}_i^B$,
  - when they have rank 2, their chain is always $\{v_{s+i}, v_i\}$.
- Solve the linear system in $\lambda_1, \ldots, \lambda_r$:
  $F = \sum_{i=1}^{r-s} v_i^{d-1}(\lambda_i v_i + \lambda_{s+i} v_{s+i}) + \sum_{i=r-s+1}^{s} \lambda_i v_i^d$.

Another example

Tangential decomposition

$F = -2x^7 - 4x^6y + 92x^6z + 15x^5y^2 - 675x^5z^2 - 20x^4y^3 + 2700x^4z^3 + 15x^3y^4 - 6075x^3z^4 - 6x^2y^5 + 7290x^2z^5 + xy^6 - 3645xz^6.$

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Check $r = 6$ and $B = \{1, y, z, y^2, z^2, y^3\}$.

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Check $r = 6$ and $B = \{1, y, z, y^2, z^2, y^3\}$.

Common eigenvectors of $(\mathbb{M}_y^B)^t$ and $(\mathbb{M}_z^B)^t$: $(1, 0, 0, 0, 0, 0)$, $(1, -1, 0, 1, 0, -1)$, $(1, 0, -3, 0, 9, 0)$.

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$F = 2x^6(x+y+z) + (x-y)^6x - 5(x-3z)^6x.$

What we can learn more and more

What we can learn more

Cactus decomposition

$$F = (x^2 + y^2 + 6xz - 8z^2)(4x - y - 5z)$$

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Start with $r = 3$,
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What we can learn more

Cactus decomposition

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What we can learn more
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Since $r/2 > 1$ there is no tg. decomposition for F with $r = 3$, i.e.

$$F \neq L_1^{3-1} L_2 + L_3^3$$

for any linear form L_1, L_2, L_3.

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for any linear form L_1, L_2, L_3.

Claim: Since we did not fill any **h**, this is the unique decomposition of F of type

$$F = L^{3-2} N$$

with N a quadratic form.
So, to recover N, it is sufficient to solve a linear system:

$$F = (ax^2 + bxy + cxz + dy^2 + eyz + fz^2)(4x - y - 5z)$$

Non-definition (yet)

A *cactus decomposition* of $F \in R_d$ is a "minimal" way of writing F as

$$F = \sum_{i=1}^{s} L_i^{d-k_i} N_i$$

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MINIMAL in which sense?

What we can learn more

Cactus decomposition

Proposition

$F = \sum_{i=1}^{s} L_i^{d-k_i} N_i$ iff $\exists\, \zeta_1, \dots, \zeta_s \in \mathbb{K}^n$, an extension $\Lambda \in R^*$ of $f^* \in R^*_{\leq d}$ and $\{p_i\}_{i \in \{1,\dots,d\}} \subseteq R$ s.t.

$$\Lambda = \sum_{i=1}^{s} \mathbb{1}_{\zeta_i} \circ p_i(\delta). \tag{1}$$

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Definition

Λ as in (1) such that

$$r = \oplus_{i=1}^{s} \underbrace{\dim_{\mathbb{K}} \langle \{ \mathbb{1}_{\zeta_i} \circ \partial^{\alpha} p_i \}_{|\alpha| \leq \deg p_i} \rangle_{\mathbb{K}}}_{:= r_i = \mathrm{mult}\, \mathbb{1}_{\zeta_i}}.$$

is minimal, is called a *generalized decomposition of f^**.

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Definition and Theorem [–,B,M]

The minimal r for which there exists a generalized decomposition of $f^* \in R_d$ is the *cactus rank* of F.

Proposition

Let $\Lambda = \sum_{i=1}^{s} \mathbb{1}_{\zeta_i} \circ p_i(\delta)$ be a generalized decomposition of f^*. Then there exist $k_i \in \mathbb{K}$, $N_i \in R_{k_i}$ such that F can be written as

$$F = \sum_{i=1}^{s} L_i^{d-k_i} N_i. \qquad (*)$$

$(*)$ is called a *cactus decomposition of* F.

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Can we recover a cactus decomposition of a given $F \in R_d$?

[BCMT] Theorem

Let $F \in R_d$.

The minim r for which $\exists$ an extension $\Lambda \in R^*$ of f^* with $\mathrm{rk} H_\Lambda = r$

$$=$$

The minimum r which allows to fill $\mathbb{H}_\Lambda(\mathbf{h})$ in order to have commuting multiplication operators.

[BCMT] Theorem

Let $F \in R_d$.

The minim r for which $\exists$ an extension $\Lambda \in R^*$ of f^* with $\mathrm{rk} H_\Lambda = r$

$$=$$

The minimum r which allows to fill $\mathbb{H}_\Lambda(\mathbf{h})$ in order to have commuting multiplication operators.

Find commuting operators $\Rightarrow$ read the $\mathbb{1}_{\zeta_i}$'s as the common rank-1 eigenvectors for the $M^t_{x_j}$.

Input: A homogeneous polynomial $F(x_0, x_1, \ldots, x_n)$ of degree d.
Output: The cactus rank of F and the L_i s.t. $F = \sum_{i=1}^{k} \lambda_i L_i^{d-k_i} N_i$ is a cactus decomposition of F.

- Construct the matrix $\mathbb{H}_\Lambda(\mathbf{h})$ with the parameters $\mathbf{h} = \{h_\alpha\}_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| > d}}$.
- Set $r := \mathrm{rk} \mathbb{H}_{f*}^{\square}$.
  1. Compute a set B of a complete staircase monomials with $|B| = r$.
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i \star \Lambda}^B (\mathbb{H}_\Lambda^B)^{-1}$ commute.
  3. If there is no solution, restart the loop with $r := r + 1$.
- Else **Output 1)** r is the cactus rank of F.
  4 Compute the eigenvectors $v_1, \ldots, v_s$ of a generic $\sum_i \alpha_i \mathbb{M}_i^B$.
- **Output 2)** $L_i := v_i$, $i = 1, \ldots, s$.

Recall

Definition

The minimal $r = \oplus_{i=1}^{s} \underbrace{\dim_{\mathbb{K}} \langle \{ \mathbb{1}_{\zeta_i} \circ \partial^{\alpha} p_i \}_{|\alpha| \le \deg p_i} \rangle_{\mathbb{K}}}_{:= r_i = \mathrm{mult}\, \mathbb{1}_{\zeta_i}}$ for which there exists a generalized decomposition of $F \in R_d$ is the *cactus rank* of F.

Recall

Definition

The minimal $r = \oplus_{i=1}^{s} \underbrace{\dim_{\mathbb{K}} \langle \{ \mathbb{1}_{\zeta_i} \circ \partial^{\alpha} p_i \}_{|\alpha| \leq \deg p_i} \rangle_{\mathbb{K}}}_{:= r_i = \operatorname{mult} \mathbb{1}_{\zeta_i}}$ for which there exists a generalized decomposition of $F \in R_d$ is the *cactus rank* of F.

The last thing that we can do is to recover each r_i.

$F \in R_d$, $\Lambda = \sum_{i=1}^{s} \mathbb{1}_{\zeta_i} \circ p_i(\delta) \in R^*$ a generalized decomposition of f^*.

Theorem

For every $j \in \{1, \ldots, n\}$ and every $\alpha \in \mathbb{N}^n$ the element

$$\mathbb{1}_{\zeta_i} \circ (\partial^\alpha p_i)(\delta) \in \mathcal{A}_\Lambda^*$$

is either the zero map or a generalized eigenvector common to every $M_{x_j}^t$ with eigenvalue $(\zeta_i)_j$.

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Corollary

If $V^j[\mu]$ is the generalized eigenspace of $M_{x_j}^t$ relative to the eigenvalue μ, then for every $i \in \{1, \ldots, s\}$ the multiplicity of $\mathbb{1}_{\zeta_i}$ is given by

$$\operatorname{mult} \mathbb{1}_{\zeta_i} = \dim_{\mathbb{K}} \cap_{j=1}^{n} V^j[(\zeta_i)_j].$$

Back to a previous example

Tangential decomposition was cactus

$F = 2x^6(x+y+z) + (x-y)^6 x - 5(x-3z)^6 x.$

Common eigenvectors of $(\mathbb{M}_y^B)^t$ and $(\mathbb{M}_z^B)^t$:

$v_1 = (1,0,0,0,0,0) \in V^y[0], V^z[0],$

$v_2 = (1,-1,0,1,0,-1) \in V^y[-1], V^z[0],$

$v_3 = (1,0,-3,0,9,0) \in V^y[0], V^z[-3].$

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$$(\mathbb{M}_z^B)^t : \underbrace{\left\langle\left\langle \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}1\\-1\\0\\1\\0\\-1\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}1\\0\\0\\-1\\0\\2\end{pmatrix}\right\rangle\right\rangle}_{V^z[0]}, \underbrace{\left\langle \begin{pmatrix}1\\0\\-3\\0\\9\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\-9\\0\end{pmatrix}\right\rangle}_{V^z[-3]}.$$

Input: A homogeneous polynomial $F(x_0, x_1, \dots, x_n)$ of degree d.
Output: All multiplicity of the $\mathbb{1}_{\zeta_i}$'s of the generalized decomposition of f^*.

- Construct the matrix $\mathbb{H}_\Lambda(\mathbf{h})$ with the parameters $\mathbf{h} = \{h_\alpha\}_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|>d}}$.
- Set $r := \mathrm{rk}\mathbb{H}_{f^*}^{\square}$.
- Compute a set B of a complete staircase monomials with $|B| = r$.
- Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i\star\Lambda}^B(\mathbb{H}_\Lambda^B)^{-1}$ commute.
- If there is no solution, restart the loop with $r := r+1$. Else r is the cactus rank of f.
- Compute the common eigenvectors $v_1, \dots, v_s$ of the $\mathbb{M}_j^B$'s and $V^j[(\zeta_1)_j]$ the generalized eigenspace of $M_{x_j}^t$ relative to the eigenvalue $(\zeta_1)_j$
- **Output** mult $\mathbb{1}_{\zeta_i} = \dim_{\mathbb{K}} \cap_{j=1}^n V^j[(\zeta_i)_j]$.

Further work

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Thanks!